

# HOW WE COUNT

or

## IS IT POSSIBLE THAT *TWO TIMES TWO IS NOT EQUAL TO FOUR*

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*[Arithmetic] is one of the oldest branches, perhaps the very oldest branch of human knowledge; and yet some of its most abstruse secrets lie close to its tritest truths.*

H. J. S. Smith

**Abstract:** People think that their counting is governed by the rules of the conventional arithmetic and that other kinds of arithmetic do not exist and cannot exist. It is demonstrated in this paper that this popular image of counting is incorrect. In many situations, we have to utilize different rules of counting. This is a result of existence of many different arithmetics. To distinct them, we call the conventional arithmetic Diophantine, while all other arithmetics we call non-Diophantine. Theory of non-Diophantine arithmetics is developed in the book of the author “Non-Diophantine arithmetics.”

From all things that people are doing, counting is one of the most important. Without counting we cannot do a lot: we cannot buy and sell, we cannot develop science and technology, we cannot organize mass production and so on and so forth. Every woman and every man, every boy and every girl perform counting many times a day. Calculators and computers were invented to help people to count. Later computer began to fulfil much more sophisticated tasks. Some of its abilities look miraculous. However, counting lies at the bottom of all computer operations.

People started using numbers and counting in prehistoric times. For thousands of years mathematicians studied numbers and counting learning a lot in this area. People's experience with numbers is profound. However, do you think that we know everything about numbers and counting?

Science and mathematics are functioning in a sequence of transformations. Sometimes these transformations change the whole foundation of human knowledge. For example, the Euclidean geometry was believed for 2200 years to be unique (both as an absolute truth and a necessary mode of human perception). People were not even able to imagine something different. The famous German philosopher Emmanuel Kant claimed that (Euclidean) geometry is given to people a priori, i.e., without special learning. In spite of this, almost unexpectedly some people began to understand that geometry is not unique. Trying to improve the axiomatic system suggested for geometry by Euclid, three great mathematicians of the 19<sup>th</sup> century (C.F.Gauss, N.I. Lobachewsky, and Ja. Bolyai) discovered a lot of other geometries. At first, even the best mathematicians opposed this discovery and severely attacked Lobachewsky and Bolyai who published their results. Forecasting such antagonistic attitude, the first mathematician of his times Gauss was afraid to publish this. Nevertheless, progress of mathematics brought understanding and then recognition. This discovery is now considered as one of the highest achievements of the human genius. It changed to a great extent understanding of mathematics and improved comprehension of the whole world.

In the 20<sup>th</sup> century, a similar situation existed in arithmetic. For thousands of years, much longer than that for the Euclidean geometry, only one arithmetic existed. Mathematical establishment treated arithmetic as primordial entity. For example, such prominent German

mathematician as Leopold Kronecker (1825-1891) wrote: "*God made the integers, all the rest is the work of man*".

Laymen have been even more persistent on this point of view. Almost all people, mathematicians as well as non-mathematicians, have had and have no doubts that  $2 + 2 = 4$  is the most evident truth in the world, which is valid always and everywhere. As it is written in the authoritative mathematical journal "The American Mathematical Monthly" (April, 1999, p.375), "*Although other sciences and philosophical theories change their 'facts' frequently,  $2 + 2$  remains 4.*"

However, in our days, some people begin to doubt the absolute character of the ordinary arithmetic, where  $2 + 2 = 4$  and two times two is equal to four. That is why scientists and mathematicians draw attention of the scientific community to the foundational problems of natural numbers and the conventional arithmetic. The most extreme view is that there is only a finite quantity of natural numbers. It is one of the central postulates of ultraintuitionism (Yesenin-Volpin, 1960). Van Danzig in his article (1956) explained why only some of natural numbers may be considered finite. Consequently, all other mathematical entities that are called traditionally natural numbers are only some expressions but not numbers. These arguments are supported and extended in (Blehman et al., 1983).

Other authors are more moderate in their criticism of the conventional arithmetic. They write that not all natural numbers are similar in contrast to the presupposition of the conventional arithmetic that the set of natural numbers is uniform (Kolmogorov, 1961; Littlewood, 1953; Birkhoff and Barti, 1970; Rashevsky, 1973; Dummett 1975; Knuth, 1976). Different types of natural numbers have been introduced, but without changing the conventional arithmetic. For example, one of the greatest mathematicians of the 20<sup>th</sup> century Andrei Kolmogorov (1961) suggested that in solving practical problems it is worth to separate *small, medium, large, and super-large* numbers.

A number  $A$  is called *small* if it is possible in practice to go through and work with all combinations and systems that are built from  $A$  elements each of which has two inlets and two outlets.

A number  $B$  is called *medium* if it is possible to go through down and work with this number. However, it is impossible to go through and work with all combinations and systems that are built from  $B$  elements each of which has two or more inlets and two or more outlets.

A number  $C$  is called *large* if it is impossible to go through a set with this number of elements. However, it is possible to elaborate a system of denotations for these elements.

If even this is impossible, then a number is called *super-large*.

According to this classification, 3, 4, and 5 are small numbers, 100, 120, and 200 are medium numbers, while an example of a large number is given by the quantity of all visible stars. Really, if we invite 4 people, we can consider all their possible positions at a dinner table. If you come to some place where there are 100 people, you can shake hands with everybody. Although, it might take too much time. What concerns the visible stars, you cannot count them, although, a catalog of such stars exists. Using this catalog, it is possible to find information about any of these stars.

This classification of numbers is based on our counting abilities. Consequently, borders between classes are vague and unstable. Higher abilities make borders between classes higher. For example, 10 is a medium number for an ordinary individual, but a small number for a computer. However, some numbers belong to one of these classes in all known situations. For example, 300 is a medium number both for people and computers.

In a similar way to what has been done by Kolmogorov and on akin grounds, the outstanding British mathematician John Edensor Littlewood (1953) separated all natural numbers into an infinite hierarchy of classes.

All mathematicians who were wise enough to distrust the complete adequacy of the conventional arithmetic may be divided into two groups. Some write (like Kolmogorov or Littlewood) that on practice natural numbers are different from those that we know from mathematics. Others admit (like Kline, 1967) that different arithmetics exist but we do not know what they are.

The well-known Russian mathematician Pyotr Rashevsky (1973) was one of the firsts to formulate explicitly the problem of construction of arithmetics that are different from the conventional one. More recently, Brian Rotman (1997) also directly formulated the problem

of elaboration of arithmetics that were essentially different from the conventional one. He based his suggestions on a series of examples demonstrating that many laws of the conventional arithmetic are not true in different situations. Rotman calls such, unimaginable for many new structures non-Euclidean arithmetics, although he does not describe them. However, it is more natural to call the conventional arithmetic by the name '*Diophantine arithmetic*' than '*Euclidean arithmetic*' because Diophantus contributed much more to its development than Euclid. Consequently, new structures acquire the name '*non-Diophantine arithmetics*'.

The main arguments of Morris Kline concerning arithmetic are similar to those from (Rashevsky, 1973; Burgin, 1977). Although, the examples supporting these arguments are different. Let us consider one of these examples, which is taken from the book of Morris Kline (1967).

If a farmer has two herds consisting of 10 and 25 heads of cows, respectively, he knows by adding 10 and 25 that the total number of cows is 35. That is, he need not count his cows. Suppose, however, he brings the two herds of cows to market where they are selling for \$100 apiece. Will a herd of 10 cows which might bring \$1000 and a herd of 25 cows which might bring \$2500 together bring in \$3500? Every businessman knows that when supply exceeds demand, the price may drop, and hence 35 cows may bring in only \$3000. In some idealized world the value of the cows may continue to be \$3500, but in actual situations this need not be true.

Consequently, continues Kline (1967), mathematicians are, of course, free to introduce the symbols 1, 2, 3, ... , where 2 means  $1+1$ , 3 means  $2+1$ , and so on. We can even deduce from this that  $2+2=4$ . But the question is not whether the mathematician can set up definitions and axioms and deduce conclusions. It is necessary to know whether this system necessarily expresses truths about the physical world.

According to Kline (1967), discovery of non-Euclidean geometries had taught mathematicians that geometry does not offer ultimate truths. That was the reason why many turned to the ordinary number system and the developments built upon it and maintained that this part of mathematics still offers unquestionable truths. The same thought is often

expressed today by people who, wishing to give an example of an absolute truth, quote  $2 + 2 = 4$ . However, examination of the relationship between our ordinary number system and the physical situations to which it is applied vividly demonstrates that it does not offer truths.

In the same way as it was with the Euclidean geometry, the conventional arithmetic has been unique and nonchallengable - other arithmetics were unknown to people. Its position in human society has been and is now even more stable and firm than the position of the Euclidean geometry before the discovery of the non-Euclidean geometries. Really, all people use arithmetic for counting. At the same time, Euclidean geometry is only studied at school and in real life rather few specialists use it. In addition to this, it is arithmetic, and not geometry, which is considered as a base for the whole mathematics in the intuitionistic and constructive approaches in mathematics.

By H.J.S.Smith, arithmetics (and namely, the Diophantine arithmetic) is one of the oldest branches, perhaps the very oldest branch, of human knowledge. His older contemporary, very talented mathematician C.O.Jacobi (1805-1851) said: "*God ever arithmetizes*".

Nevertheless, in spite of such a high estimation of the conventional arithmetic, its uniqueness and indisputable authority has been recently challenged. The first family of other, essentially different arithmetics was discovered in 1975 (Burgin, 1977). Having many arithmetics, we need a special name for the conventional one because it occupies a unique place in the family of all arithmetics. The conventional arithmetic may be called Diophantine because the ancient Greek mathematician Diophantus was the first who made an essential contribution to arithmetic. Consequently, other arithmetics are called non-Diophantine arithmetics.

Like geometries of Lobachewsky, non-Diophantine arithmetics from the first family, which are called *projective arithmetics*, depend on a special parameter. However, this parameter is not a number as in the case of Lobachewsky geometries. Projective arithmetics have a functional parameter. The Diophantine arithmetic is a member of this family: its parameter is equal to the identity function  $f(x) = x$ .

Later another family of non-Diophantine arithmetics, which are called *dual arithmetics*, was introduced in (Burgin, 1980). This family of arithmetics also has a functional parameter,

and the Diophantine arithmetic is a member of this new family: its parameter is equal to the identity function  $f(x) = x$ . However, many properties of the arithmetics from the second family are essentially different in comparison with arithmetics from the first family. For example, in projective arithmetics, we can have an identity  $n + 1 = n$ . In dual arithmetics,  $n + 1$  is always larger than  $n$ . The book "*Non-Diophantine Arithmetics*" (Burgin, 1997) contains a detailed and extended study of these arithmetics.

In some non-Diophantine arithmetics, even the most evident truth (such as  $2 + 2 = 4$  or two times two is equal to four) may be discarded. Some of them (projective arithmetics) possess similar properties to those of transfinite numbers arithmetics built by the great German mathematician Georg Cantor. For example, a non-Diophantine arithmetic may have a sequence of numbers  $a_1, a_2, \dots, a_n, \dots$  such that for any number  $b$  that is less than some  $a_n$  the equality  $a_n + b = a_n$  is valid. It is also possible that two times two is not equal to four. There are such non-Diophantine arithmetics that have the largest number.

However, it is so hard for people to understand and what is even more difficult to accept non-Diophantine arithmetics. Power of people's stereotypes is vividly demonstrated by the book (Blehman *et al.*, 1983). At first (section 1.2.4), the authors of the book explain with many examples and references that our intuition of natural numbers and arithmetic are very misleading. However, after this (section 1.2.5), they announce that it is completely impossible that two times two is not equal to four. The authors are even trying to prove this utilizing the following probabilistic reasoning. Here are their arguments (p.50):

*Really, the statement that two times two is equal to four may be taken as an example of the most evident truth. Although, nobody doubts that this is a true equality, it is possible to evaluate formally probability that in reality two times two is equal to five, while the standard statement that two times two is equal to four is a result of a constantly repeated arithmetical mistake. Let us suppose that any individual performing multiplication with numbers that are less than ten can decrease the result by one with the probability  $10^{-6}$ . This corresponds to several such mistakes during his or her life. If we assume that that through the whole history of mankind,  $10^{10}$  people performed the multiplication "two times two"  $10^6$  times during the life of each of them, then the probability that they repeated this mistake of decreasing the*

result is less than  $10^{-10}$ <sup>17</sup>. Thus, the authors conclude, *the probability is so small that the event is absolutely impossible and we see that two times two is equal to four.*

However, there are such non-Diophantine arithmetics in which two times two is not equal to four. Besides, the authors are absolutely sure that nobody ever did the mistake. Both these facts show to what extent probabilistic proofs may be misleading. This reminds us how some people try to prove that it is impossible that life emerged by itself. Actually, they only calculate some small probability of this event. Then, in spite that some of their assumptions are rather dubious, they claim that all living organisms were created.

When we consider non-Diophantine arithmetics, it is possible to think that they are absolutely formal constructions like many other mathematical objects, which are very far from the real world. But let us recollect that a similar skepticism and mistrust met the discovery of non-Euclidean geometries. Even Gauss (in spite of being acknowledged as the greatest mathematician of his time) did not dare to publish his results concerning these geometries as he was not able to find anything that is similar to them in nature. Lobachewsky called his geometry an imaginable one. But afterwards it was discovered that the real physical space fits non-Euclidean geometries, and that the Euclidean geometries do not have such essential applications as the non-Euclidean ones. In this aspect, the situation with non-Diophantine arithmetics is different. In spite of the short time, which has passed after their discovery, it has been demonstrated that many real phenomena and processes exist that match the non-Diophantine arithmetics.

As a matter of fact, much earlier than non-Diophantine arithmetics appeared, Littlewood (1953) considered an example demonstrating how the rules of non-Diophantine arithmetics (in spite of that they were unknown at that time) can be imposed upon the real world. Several similar and even more lucid examples are considered in (Davis and Hersh, 1986) and in (Kline, 1967). Let us consider some of such situations.

1. A market sells a can of tuna fish for \$1.05 and two cans for \$2.00. So, we have  $a + a \neq 2a$ .

2. In a similar way, coming to a supermarket, you may buy one gallon of milk for \$3.50, while two gallons of the same milk will cost you only \$5.50. Once more, we have  $a + a \neq 2a$ .

3. Even more, coming to a supermarket, you can see an advertisement "Buy one, get one free." Such advertisement may refer almost to any product: bread, milk, juice etc. Thus, if one gallon of orange juice costs \$5, then we have the equality  $5 + 5 = 5$ . It is impossible in the conventional arithmetic but it is true for some non-Diophantine arithmetics.

4. To make the situation, when ordinary addition is inappropriate, more explicit, an absurd but not unrelated question is formulated: If the Mona Lisa painting is valued at \$10,000,000, what would be the value of two Mona Lisa paintings.

5. Another example: when a cup of milk is added to a cup of popcorn then only one cup of mixture will result because the cup of popcorn will very nearly absorb a whole cup of milk without spillage. So, in the last case we have  $1 + 1 = 1$ . It is impossible in the conventional arithmetic but it is true for some non-Diophantine arithmetics.

These examples demonstrate that non-Diophantine arithmetics are important for business and economics. Some economical problems and inconsistencies caused by the conventional arithmetic are considered in (Tolpygo, 1997). Utilization of non-Diophantine arithmetics eliminates those problems and inconsistencies.

In addition to this, non-Diophantine arithmetics solve some problems that remained unsolved from the time of ancient Greece. One of such problems is discussed in (Rashevsky, 1973). This is the so-called, "paradox of a heap," which is attributed to Zeno.

Let us consider a heap of grains. If we add to this heap one grain, the heap is not changing. Consequently, if we take the number  $K$  of the grains in this heap, then adding 1 to  $K$  does not change  $K$ . This contradicts the main law of the Diophantine arithmetic stating that for an arbitrary number  $k$ ,  $k + 1$  is not equal to  $k$ , and gives birth to a paradox if we have only one arithmetic, which is our conventional Diophantine arithmetic.

You may ask what for we are trying to do something with puzzles that were suggested thousands years ago. However, the paradox of a heap has a direct analogy in our times. For example, you are buying a car for \$30,000. Then suddenly when you have to pay the price is changed and becomes one cent greater. Do you think that the new price is different from the initial one or consider it practically one and the same price? It is natural to suppose that any sound person has the second opinion. Consequently, we come to the same paradox that  $k + 1$  is not equal to  $k$ , where  $k$  is the price of the car in cents. As we have seen, non-Diophantine arithmetics solve this paradox.

Moreover, an interesting peculiarity of the new arithmetics is their ability to provide means for mathematical grounding of some intuitive constructions used by physicists. As an example, we can take such relations as "much bigger" (denoted by  $\gg$ ) and "much lesser" (denoted by  $\ll$ ) which are formalized in non-Diophantine arithmetics. These relations are very frequently used in physics but there they have no exact meaning without non-Diophantine arithmetics.

Other peculiarities of physical theories become clearer in the light of non-Diophantine arithmetics. For example, in physics there are some absolute values like the speed of light  $C$  or the absolute zero temperature. According to relativity theory,  $C$  is the largest speed attainable by material bodies. If you add some speed  $V$  to  $C$ , you have the same  $C$ , in other words,  $C + V = C$ . This equality is possible in some non-Diophantine arithmetics but impossible in the conventional, Diophantine arithmetic.

Moreover, finiteness of all material bodies in the universe suggests that temperature has not only the lower bound but also the upper bound, and we again come to non-Diophantine arithmetics.

That is why, some physicists (Zeldovich et al, 1990) emphasized that fundamental problems of modern physics are dependent on our ways of counting.

Other examples of situations for which non-Diophantine arithmetics give more adequate models than the Diophantine arithmetic are related to the concept of convergence for series. As it was emphasized by the great French mathematician Henri Poincare (cf. Blehman *et al.*, 1983), convergence for a physicist and a mathematician does not coincide. Really, let us

consider a simple example of two series:  $1000^n/n!$  and  $n!/1000^n$ . Mathematicians will call the first series convergent and the second series divergent because its member can grow without limits. At the same time, astronomers will say that the first series is divergent because its first 1000 member are increasing, while the second series is convergent because its first 1000 member are decreasing and rather fast.

Poincare concludes that both opinions are legal: the first one in theoretical studies and the second one in practical applications. It is important to separate these areas.

In other words, a series that converges in the sense of mathematical analysis may be divergent from the point of view of an astronomer and vice versa. However, if we have only one arithmetic, then convergence has to be unique by its definition. When, on the contrary, we have different arithmetics, then one and same definition of convergence may give unlike results. The reason for these differences is that series and their convergence are considered in different arithmetics. It is possible to show that arithmetics that are used for various applications are, as a rule, non-Diophantine. Only in mathematics, we can limit ourselves to one arithmetic simply by accepting one more postulate: *there are no other arithmetics besides the Diophantine arithmetic*. However, as we can see, this postulate is not true for reality.

Thus, we see that non-Diophantine arithmetics provide a mathematical base for a variety of ideas and situations in different areas: in science as well as in everyday life.

## References

1. Blehman, I.I., Myshkis, A.D., and Panovko, Ya.G. *Mechanics and Applied Logic*, Nauka, Moscow, 1983 (in Russian)
2. Birkhoff, G., and Bartee, T.C. *Modern Applied Algebra*, McGraw Hill, New York, 1967
3. Burgin, M.S. *Non-classical Models of Natural Numbers*, Russian Mathematical Surveys, 1977, v.32, No. 6, pp.209-210 (in Russian)
4. Burgin, M.S. *Dual Arithmetics*, Abstracts presented to the American Mathematical Society, 1980, v.1, No. 6

5. Burgin, M.S. Infinite in Finite or Metaphysics and Dialectics of Scientific Abstractions, *Philosophical and Sociological Thought*, 1992, No. 8, pp.21-32  
(in Russian and Ukrainian)
6. Burgin, M.S. *Non-Diophantine Arithmetics*, Ukrainian Academy of Information Sciences, Kiev, 1997 (in Russian)
7. Burgin, M.S. Finite and Infinite, in "*On the Nature and Essence of Mathematics*, Appendix," Kiev, 1998, pp. 97-108 (in Russian)
8. van Dantzig, D. Is  $10^{10}$  a finite number? *Dialectica*, 1956, No. 9
9. Davis, Philip J. and Hersh, Reuben, *The mathematical experience*, Houghton Mifflin Co., Boston, Mass., 1986
10. Dummett, M. Wang's paradox, *Synthese*, 1975, v. 30, No. 3-4, pp. 301-324
11. Kline, M. *Mathematics for Nonmathematicians*, Dover Publ., New York, 1967
12. Knuth, D.E. Mathematics and computer science: coping with finiteness, *Science*, 1976, v.194, No. 4271, pp. 1235-1242
13. Kolmogorov, A.N. Automata and Life, in: "*Knowledge is Power*", 1961, No. 10; No. 11
14. Littlewood, J.E. *Miscellany*, Methuen, London, 1953
15. Rashevsky, P.K. On the Axioms of Natural Numbers, *Russian Mathematical Surveys*, 1973, v.28, No. 4, pp. 243-246
16. Rotman, B. The Truth about Counting, *The Sciences*, 1997, No. 11, pp. 34-39
17. Tolpygo, A. Finite Infinity, in "*Methodological and Theoretical Problems of Mathematics and Information Sciences*," Kiev, Ukrainian Academy of Information Sciences, 1997, pp.35-44 (in Russian)
18. Wilson, A.M. *The infinite in finite*, Oxford University Press, 1996
19. Yesenin-Volpin, A.C. On the Grounding of Set Theory, In: "*Application of Logic in Science and Technology*", Moscow, 1960, pp. 22-118 (in Russian)
20. Zeldovich, Ya. B., Ruzmaikin, A.A., and Sokoloff D.D. *The Almighty Chance*, World Scientific Lecture Notes, v. 20, World Scientific, Singapore/New Jersey, 1990