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Elements of Non-Diophantine Arithmetics

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Abstract:

Science and mathematics help people to better understand world, eliminating many fallacies and misconceptions. One of such misconceptions is related to arithmetic, which is extremely important both for science and everyday life. People think that their counting is governed by the rules of the conventional arithmetic and that other kinds of arithmetic do not exist and cannot exist. However, this popular image of the situation with integer numbers is wrong. In many situations, we have to utilize different rules of counting and operating. This is a consequence of the existing diversity in nature and society and to correctly represent this diversity, people have to utilize different arithmetics. To distinct them, we call the conventional arithmetic Diophantine, while other arithmetics are called non-Diophantine. In this work, a class of non-Diophantine arithmetics called projective arithmetics is constructed and studied.

Keywords: arithmetic; operation; addition; multiplication; larger; much larger

1. Introduction

From all things that people are doing, counting is one of the most important. Without counting we cannot do a lot: we cannot buy and sell, we cannot develop science and technology, we cannot organize mass production and so on and so forth. Every woman and every man, every boy and every girl perform counting many times a day. Calculators and computers were invented to help people to count. Later computer began to fulfill much more sophisticated tasks. Some of its abilities look miraculous. However, counting lies at the bottom of all computer operations.

People started using numbers and counting in prehistoric times. For thousands of years mathematicians studied numbers and counting learning a lot in this area. People's experience with numbers is profound but incomplete.

The situation with arithmetic now is similar to the situation with geometry in the middle of the 19th century. Namely, the Euclidean geometry was believed for 2200 years to be unique (both as an absolute truth and a necessary mode of human perception). People were not even able to imagine something different. The famous German philosopher Emmanuil Kant claimed that (Euclidean) geometry is given to people *a priory*, i.e., without special learning. In spite of this, almost unexpectedly some people began to understand that geometry is not unique. Trying to improve the axiomatic system suggested for geometry by Euclid, three great mathematicians of the 19th century (C.F.Gauss, N.I. Lobachewsky, and Ja. Bolyai) discovered a lot of other, non-Euclidean geometries. At first, even the best mathematicians opposed this discovery and severely attacked Lobachewsky and Bolyai who published their results. Forecasting such antagonistic attitude, the first mathematician of his times Gauss was afraid to publish his results on non-Euclidean geometry. Nevertheless, progress of mathematics brought understanding and then recognition. This discovery is now considered as one of the highest achievements of the human genius. It changed to a great extent understanding of mathematics and improved comprehension of the whole world.

The situation with arithmetic is even more striking. For thousands of years, much longer than that for the Euclidean geometry, people explicitly used only one arithmetic. Mathematical establishment treated arithmetic as primordial entity. For example, such prominent German mathematician as Leopold Kronecker (1825-1891) wrote: "*God made the integers, all the rest is the work of man*". Carl Friedrich Gauss (1777-1855) claimed that " God does arithmetic."

Almost all people, mathematicians, as well as non-mathematicians, have had and have no doubts that 2 + 2 = 4 is the most evident truth in the world, which is valid always and everywhere. As it is written in the authoritative mathematical journal “The American Mathematical Monthly” (April, 1999, p.375), "*Although other sciences and philosophical theories change their 'facts' frequently*, 2 + 2 *remains* 4."

However, sages of the ancient Greece started to doubt convenience of the conventional arithmetic. There was a group of philosophers, who were called Sophists and lived from the second half of the fifth century B.C.E. to the first half of the fourth century B.C.E.. Sophists asserted relativity of human knowledge and suggested many paradoxes, explicating complexity and diversity of real world. The famous philosopher Zeno of Elea (490-430 B.C.E.), who was said to be a self-taught country boy, invented very impressive paradoxes, in which he questioned the popular knowledge and intuition related to such fundamental essences as time, space, and number.

However, Greek sages posed questions, but in many cases, including arithmetic, suggested no answers. As a result, for more than two thousand years these problems were forgotten and everybody was satisfied with the conventional arithmetic. In spite of all problems and paradoxes, this arithmetic has remained very and very useful.

May be the first who questioned absolute validity of the conventional arithmetic in modern times was the famous German scientist Herman Ludwig Ferdinand von Helmholtz (1821-1894). In his “Counting and Measuring” (1887), Helmholtz considered an important problem of applicability of arithmetic, at that time it was only one arithmetic, to physical phenomena. This was a natural approach of a scientist, who even mathematics judged by the main criterion of science – observation and experiment.

His first observation was that as the concept of number is derived from some practice, usual arithmetic is applicable to these experiences. However, it is easy to find many situations when this is not true. To mention but a few described by Helmholtz, one raindrop added to another raindrop does not make two raindrops. In a similar way, when one mixes two equal volumes of water, one at 40° Fahrenheit and the other at 50°, one does not get two volumes at 90°. Alike the conventional arithmetic fails to describe correctly the result of combining gases or liquids by volumes. For example (Kline, 1980), one quart of alcohol and one quart of water yield about 1.8 quarts of vodka.

Later the famous French mathematician Henri Lebesgue facetiously pointed out (cf. (Kline, 1980)) that if one puts a lion and a rabbit in a cage, one will not find two animals in the cage later on.

However, very few (if any) paid attention to the work of Helmholtz on arithmetic, and as still no alternative to the conventional arithmetic has been suggested, these problems were once more forgotten. Only much later, in the second part of the 20th century, mathematicians began to doubt once more the absolute character of the ordinary arithmetic, where 2 + 2 = 4 and two times two is equal to four. Scientists and mathematicians draw attention of the scientific community to the foundational problems of natural numbers and the conventional arithmetic. The most extreme view that there is only a finite quantity of natural numbers had Yesenin-Volpin (1960), who developed such mathematical direction as ultraintuitionism and took this assertion as one of the central postulates of ultraintuitionism. Van Danzig had similar ideas but expressed in different way. In his article (1956), he argued that only some of natural numbers may be considered finite. Consequently, all other mathematical entities that are called traditionally natural numbers are only some expressions but not numbers. These arguments are supported and extended by Blehman, *et al* (1983).

Other authors are more moderate in their criticism of the conventional arithmetic. They write that not all natural numbers are similar in contrast to the presupposition of the conventional arithmetic that the set of natural numbers is uniform (Kolmogorov, 1961; Littlewood, 1953; Birkhoff and Barti, 1970; Rashevsky, 1973; Dummett 1975; Knuth, 1976). Different types of natural numbers have been introduced, but without changing the conventional arithmetic. For example, Kolmogorov (1961) suggested that in solving practical problems it is worth to separate *small, medium, large,* and *super-large* numbers. Rashevsky (1973) explicitly formulated the problem of constructing a new arithmetic of natural numbers.

Such new arithmetics were discovered in 1975 and published in (Burgin, 1977). The conventional arithmetic may be called Diophantine because the ancient Greek mathematician Diophantus was the first who made an essential contribution to arithmetic. Thus, it is natural to call new arithmetics non-Diophantine. Different properties of non-Diophantine arithmetics and some applications are described in (Burgin, 1992; 1997; 1998; 2001).

Like non-Euclidean geometries of Lobachewsky, non-Diophantine arithmetics depend on a special parameter, although this parameter is not a number as in the case of Lobachewsky geometries. Arithmetics have a functional parameter, that is, any arithmetic in this family, properties and laws of its operations depend on a definite function $f(x)$. The Diophantine arithmetic is a member of this parametric family: its parameter is equal to the identity function $f(x) = x$.

Denotations

Let **N** be the arithmetic of all natural numbers with 0 while $+$, $\cdot$ and $\leq$ be operations of addition and multiplication, and the natural order relation in **N,** correspondingly. The set of all natural numbers is denoted by N. The set of all real numbers is denoted by R. If X is a set, then $1_X : X \rightarrow X$ is the identity mapping, i.e., $1_X(x) = x$. If n is a natural (whole) number, then Sn denotes the next number $n + 1$.

2. Projective Arithmetics

At first, we define pre-arithmetics. Informally, a *pre-arithmetic* is a subset of all natural numbers with, at least, two operations + (addition) and ◦ (multiplication), which are defined for all its elements. In what follows, an arbitrary pre-arithmetic is denoted by $\boldsymbol{A} = (A; + , \circ)$ where A is a subset of N and is called the set of the elements or numbers of $\boldsymbol{A}$. An *arithmetic* is a pre-arithmetic such that satisfies extra conditions. One of such conditions essential for applications is the linearity of the order of the numbers (i.e., elements) of the arithmetic. As an example of application where this condition is important, it is possible to take utility function used for decision making. However, non-Diophantine arithmetics are not defined axiomatically (by this or any other conditions). They are constructed by the procedure that is described below, using the conventional natural numbers.

In what follows, an arbitrary arithmetic is denoted by $\boldsymbol{A} = (A ; + , \circ , \leq)$. The above condition means that all numbers in $\boldsymbol{A}$ have to be linearly ordered by the relation ≤ .

Let us take two pre-arithmetics $\boldsymbol{A}_1 = (A_1 ; +_1 , \circ_1)$ and $\boldsymbol{A}_2 = (A_2 ; +_2 , \circ_2)$ as well as two functions $g: A_1 \to A_2$ and $h: A_1 \to A_2$.

Definition 1. A *pre-arithmetic* $\boldsymbol{A}_1 = (A_1 ; +_1 , \circ_1)$ is called weakly projective (projective) with respect to a pre-arithmetic $\boldsymbol{A}_2 = (A_2 ; +_2 , \circ_2)$ if ($hg = 1_{A1}$ and $hg = 1_{A2}$) the two operations in $\boldsymbol{A}_1$ are defined for any two numbers a and b as follows:

$$a +_1 b = h (g(a) +_2 g(b)) ;$$

$$a \circ_1 b = h (g(a) \circ_2 g(b)) .$$

Definition 2. The function g is called the *projector* and the function h is called the *coprojector* for the pair $(\boldsymbol{A}_1 , \boldsymbol{A}_2)$.

If we have a set B, a pre-arithmetic $\boldsymbol{A} = (A ; + , \circ)$ and two functions $g: B \to A$ and $h: A \to B$, then it is possible to define on B the structure of the pre-arithmetic that is weakly projective with respect to $\boldsymbol{A}$. It will be a unique pre-arithmetic with the projector g and the coprojector h.

Theorem 1. If a pre-arithmetic $\boldsymbol{A}_1 = (A_1 ; +_1 , \circ_1)$ is weakly projective with respect to a pre-arithmetic $\boldsymbol{A}_2 = (A_2 ; +_2 , \circ_2)$ and $gh = 1_{A2}$, then the pre-arithmetic $\boldsymbol{A}_2$ is projective with respect to a pre-arithmetic $\boldsymbol{A}_1$.

Let us consider an arbitrary non-decreasing function $f: U \to R$ defined on a subset U of R. It is possible to define the following two functions $f_{\mathrm{T}}: M \to N$ and $f^{\mathrm{T}}: P \to M$ where M , P are some subsets of N.

$$f_{\mathrm{T}}\,(x) = \inf \{n;\ n \in N \text{ and } n \geq f(x)\ \} = \,]f(x)[\, = \lceil f(x) \rceil ;$$

$$f^{\mathrm{T}}\,(x) = \sup \{m;\ m \in N \text{ and } m \leq f^{-1}(x)\ \} = [f^{-1}(x)] = \lfloor f^{-1}(x) \rfloor \ .$$

Definition 3. A pre-arithmetic $\boldsymbol{A} = (M ; + , \circ)$ is called a *projective pre-arithmetic* if it is weakly projective with respect to the conventional (Diophantine) arithmetic $\mathbf{N} = (N ; + , \cdot , \leq)$ with the *projector* $f_{\mathrm{T}}(x)$ and the *coprojector* $f^{\mathrm{T}}(x)$. The function $f(x)$ is called the *generator* of the *projector* $f_{\mathrm{T}}(x)$, *coprojector* $f^{\mathrm{T}}(x)$ and the pre-arithmetic $\boldsymbol{A}$.

Operations in $\boldsymbol{A}_1$ are defined for any two numbers a and b as follows:

$$a + b = f^{\mathrm{T}}\,(f_{\mathrm{T}}\,(a) + f_{\mathrm{T}}\,(b)\,) ;$$

$$a \circ b = f^{\mathrm{T}}\,(f_{\mathrm{T}}\,(a) \cdot f_{\mathrm{T}}\,(b)\,) .$$

Elements of M are called numbers of $\boldsymbol{A}$ and are denoted with subscript μ, i.e., 2 in $\boldsymbol{A}$ is denoted by 2_{μ} and 5 in $\boldsymbol{A}$ is denoted by 5_{μ} . Numbers of $\boldsymbol{A}$ are ordered by the same order relation ≤ as they are ordered in **N.**

Example 1. Let us take the set $M = \{1, 2, \dots , m\ \}$ and define the projector $f_{\mathrm{T}}: M \to N$ as the natural inclusion of M into N . The coprojector $f^{\mathrm{T}}: N \to N$ is defined by the formula $f^{\mathrm{T}}(n) = k$ where $1 \leq k < m$ and $n - k$ is divisible by m. Then the projective pre-arithmetic $\boldsymbol{A} = (M ; + , \circ)$ defined by these functions is the arithmetic of residues modulo m.

Let $\boldsymbol{A} = (M ; + , \circ)$ be a projective pre-arithmetic.

Theorem 2. If $f_T\ (0_\mu) = 0$, then for any $a_\mu \in \boldsymbol{A}$ the equality $0_\mu + a_\mu = a_\mu$ is true if and only if $f\,(x)$ is a strictly increasing function.

This theorem gives necessary and sufficient conditions for 0_μ to be a neutral element (zero) with respect to addition in the projective pre-arithmetic $\boldsymbol{A}$.

Proposition 1. If $f_T\ (0_\mu) = 0$, then for any $a_\mu \in \boldsymbol{A}$ the equality $0_\mu \circ a_\mu = a_\mu$ is true.

It means that the natural equality $f_T\ (0_\mu) = 0$ always imply that 0_μ is a absorbing element (zero) with respect to multiplication in the projective pre-arithmetic $\boldsymbol{A}$.

It implies the following definition.

Definition 4. A projective pre-arithmetic $\boldsymbol{A} = (\ N\ ; +\ , \circ, \leq\ )$ with the projector $f_T\ (x)$ and the coprojector $f^{\,T}\ (x)$ is called a *projective arithmetic* if the following conditions are satisfied:

1) $f_T\ (0_\mu) = 0$;

2) $f\,(x)$ is a strictly increasing function;

3) for any elements a and b from U from $a \leq b$, we have $f_T\ (Sa) - f_T\ (a) \leq f_T\ (Sb) - f_T\ (b)$.

Here, Sa is the element of $\boldsymbol{A}$ that strictly follows a.

Thus, operations in $\boldsymbol{A}$ are defined for any two its numbers a_μ and b_μ as follows:

$$a_\mu + b_\mu = f^{\,T}\ (f_T\ (a) + f_T\ (b)\ )\ ;$$

$$a_\mu \circ b_\mu = f^{\,T}\ (f_T\ (a) \cdot\ f_T\ (b)\ )\ .$$

Besides, a natural order relation is defined on A:

$$a_\mu \leq b_\mu \ \text{ if and only if } \ a \leq b\ .$$

Remark 1. It is possible to define the generator (function f) only for natural numbers, but in many cases, its definition for real numbers makes the analytic expression for f simpler.

Example 2. Let us take as a generator $f(x)$ for a projective arithmetic $\boldsymbol{A} = (\ N\ ; +\ , \circ, \leq)$ a simple function, such as x^2, and look how operations are performed.

$2_\mu + 2_\mu = f^T (f_T (2_\mu) + f_T (2_\mu)) = f^T (4 + 4) = f^T (8) = \sup \{m;\ m \in N$ and $m \le \sqrt{8}\ \} = 2_\mu$ because $f^T (3) = 9 > 8$ or $3 > \sqrt{8}$.

$2_\mu + 3_\mu = f^T (f_T (2_\mu) + f_T (3_\mu)) = f^T (4 + 9) = f^T (13) = \sup\{m;\ m \in N$ and $m \le \sqrt{13}\} = 3_\mu$ because $f^T (4) = 16 > 13$ or $4 > \sqrt{13}$.

$10_\mu + 11_\mu = f^T (f_T (10_\mu) + f_T (11_\mu)) = f^T (100 + 121) = f^T (221) = \sup \{m;\ m \in N$ and $m \le \sqrt{221}\ \} = 14_\mu$ because $f^T (15) = 225 > 221$ and $f^T (14) = 196 < 221$.

In a similar way, we find that:

$2_\mu + 11_\mu = 11_\mu$, $3_\mu + 11_\mu = 11_\mu$, $4_\mu + 11_\mu = 11_\mu$, but $5_\mu + 11_\mu = 12_\mu$, $6_\mu + 11_\mu = 12_\mu$, $7_\mu + 11_\mu = 13_\mu$, $8_\mu + 11_\mu = 15_\mu$, and $11_\mu + 11_\mu = 15_\mu$.

$2_\mu \circ 2_\mu = f^T (f_T (2_\mu) \cdot f_T (2_\mu)) = f^T (4 \cdot 4) = f^T (16) = 4_\mu$.

$2_\mu \circ 3_\mu = f^T (f_T (2_\mu) \cdot f_T (3_\mu)) = f^T (4 \cdot 9) = f^T (36) = 6_\mu$.

In general, we have $n^2 \cdot m^2 = (n \cdot m)^2$. Consequently, in this arithmetic $\boldsymbol{A}$, multiplication of numbers is the same as in the Diophantine arithmetic, i.e., $n_\mu \circ m_\mu = (n \cdot m)_\mu$, while addition is essentially different. As we have seen, two times two is still four, while two plus two is only two.

Example 3. Let us take as a generator for a projective arithmetic $\boldsymbol{A} = (N ; + , \circ, \le)$ a simple function, such as $10x$.

$2_\mu + 2_\mu = f^T (f_T (2_\mu) + f_T (2_\mu)) = f^T (20 + 20) = f^T (40) = 4_\mu$.

$2_\mu + 3_\mu = f^T (f_T (2_\mu) + f_T (3_\mu)) = f^T (20 + 30) = f^T (50) = 5_\mu$.

This is a general case for addition of numbers in $\boldsymbol{A}$ as $n_\mu + m_\mu = f^T (f_T (n_\mu) + f_T (m_\mu)) = f^T (10n + 10m) = f^T (10(n + m)) = (n + m)_\mu$.

At the same time, we have:

$2_\mu \circ 2_\mu = f^T (f_T (2_\mu) \cdot f_T (2_\mu)) = f^T (20 \cdot 20) = f^T (400) = 40_\mu$.

$2_\mu \circ 3_\mu = f^T (f_T (2_\mu) \cdot f_T (3_\mu)) = f^T (20 \cdot 30) = f^T (600) = 60_\mu$.

This is a general case for addition of numbers in $\boldsymbol{A}$ as $n_\mu \circ m_\mu = f^T (f_T (n_\mu) \cdot f_T (m_\mu)) = f^T (10n \cdot 10m) = f^T (10(n \cdot m)) = (n \cdot m)_\mu$.

In this arithmetic ***A***, addition of numbers is the same as in the Diophantine arithmetic, while multiplication is essentially different. As we have seen, two plus two is still four, while two times two is equal to forty.

Theorem 3. Both operations, addition + and multiplication $\circ$, are commutative in any projective arithmetic.

Remark 2. In some projective arithmetics, addition + or/and multiplication $\circ$ can be non-associative.

There are other operations in ***A***. For instance, we have two *n*-ary operations:

1. $\Sigma^{n} (a_{\mu,1} , a_{\mu, 2}, \dots , a_{\mu,n-1} , a_{\mu,n}) = f^{T} (f_{T} (a_{\mu,1}) + f_{T} (a_{\mu, 2}) + \dots + f_{T} (a_{\mu, n}))$;
2. $\Pi^{n} (a_{\mu,1} , a_{\mu, 2}, \dots , a_{\mu,n-1} , a_{\mu,n}) = f^{T} (f_{T} (a_{\mu,1}) \cdot f_{T} (a_{\mu, 2}) \cdot \dots \cdot f_{T} (a_{\mu, n}))$.

Example 4. Let us take a projective arithmetic $\boldsymbol{A} = (N ; + , \circ, \leq)$ with the generator $f(x) = x^2$, and look how *n*-ary operations are performed.

$\Sigma^{n} (2_{\mu} , 2_{\mu} , 2_{\mu}) = f^{T} (4 + 4 + 4) = f^{T} (12) = 3_{\mu}$.

$\Sigma^{n} (2_{\mu} , 2_{\mu} , 3_{\mu}) = f^{T} (4 + 4 + 9) = f^{T} (17) = 4_{\mu}$, but

$\Sigma^{n} (1_{\mu} , 1_{\mu} , 1_{\mu} , 1_{\mu} , 1_{\mu} , 3_{\mu}) = f^{T} (1 + 1 + 1 + 1 + 1 + 9) = f^{T} (14) = 3_{\mu}$.

$\Pi^{n} (2_{\mu} , 5_{\mu} , 8_{\mu}) = f^{T} (4 \cdot 25 \cdot 64) = f^{T} (6400) = 80_{\mu} = (2 \cdot 5 \cdot 8)_{\mu}$.

This is a general case for the operation Π^{n} as $m_1^{\ 2} \cdot m_1^{\ 2} \cdot \dots \cdot m_n^{\ 2} = (m_1 \cdot m_1 \cdot \dots \cdot m_n)^2$.

In the Diophantine arithmetic, we have $\Sigma^{n} (a_1 , a_2, \dots , a_{n-1} , a_n) = a_1 + a_2 + \dots + a_n$. However, in some projective arithmetics, addition is not associative, and we do not have the corresponding identity $\Sigma^{n} (a_{\mu,1} , a_{\mu, 2}, \dots , a_{\mu,n-1} , a_{\mu,n}) = a_{\mu,1} + a_{\mu, 2} + \dots + a_{\mu, n}$. Thus, it is an important question in what projective arithmetics addition is associative.

Theorem 4. Addition + in a projective arithmetic $\boldsymbol{A} = (M ; + , \circ, \leq)$ is associative if and only if the function *f* is piecewise linear.

In addition to ≤ and <, there are other kinds of order relations in ***A***:

1) $a_\mu << b_\mu$ means that a_μ is *much less* than b_μ and in this case, b_μ is *much larger* than a_μ :

$$a_\mu << b_\mu \text{ if and only if } b_\mu + a_\mu = b_\mu .$$

2) $a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1} <<_n b_\mu$ means that the group $a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1}$ is *much less* than b_μ and in this case, b_μ is *much larger* than the group $a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1}$:

$$a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1} <<_n b_\mu \text{ if and only if } \Sigma^{n} (a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1}, b_\mu) = b_\mu..$$

3) $a_\mu <<< b_\mu$ means that a_μ is *much much less* than b_μ and in this case, b_μ is *much much larger* than a_μ :

$$a_\mu <<< b_\mu \text{ if and only if } b_\mu \circ a_\mu = b_\mu..$$

4) $a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1} <<<_n b_\mu$ means that the group $a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1}$ is *much much less* than b_μ and in this case, b_μ is *much much larger* than the group $a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1}$:

$$a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1} <<<_n b_\mu \text{ if and only if } \Pi^{n} (a_{\mu,1}, a_{\mu,2}, \dots, a_{\mu,n-1}, b_\mu) = b_\mu..$$

It is possible that << is a total relation on *M*.

Example 5. Let us take $f_T(n) = 2^{2^n}$ as a generator for a projective arithmetic.

Then for any *n*, we have

$$f_T(n+1) + f_T(n) = 2^{2^{n+1}} + 2^{2^{n}} < 2^{2^{n+1}} + 2^{2^{n+1}} < 2^{2^{n+1}} \cdot 2^{2^{n+1}} = 2^{2^{n+1} + 2^{n+1}} = 2^{2 \cdot 2^{n+1}} = 2^{2^{n+2}} = f_T(n+2)$$

Consequently, we have the following relations $1_\mu << 2_\mu << 3_\mu << \dots << n_\mu << (n+1)_\mu << \dots$ in the projective arithmetic with the projector $f_T(n)$.

Theorem 5. Relation << in a projective arithmetic ***A*** = (*M* ; + , ◦, ≤) is transitive and disjunctively asymmetric, i.e., only one relation *x*Q*y* or *y*Q*x* is valid for all different elements *x* and *y* from *M*.

The proof is based on the following lemma.

Lemma 1. For any number *a* from a projective arithmetic ***A***, we have $a << Sa$ if $f_T(Sa) + f_T(a) < f_T(SSa)$.

Definition 4. a) A binary relation P on a set X is called *compatible from the right* (*from the left*) with a binary relation Q on X if $P \circ Q \subseteq P$ ($Q \circ P \subseteq P$).

b) A binary relation P on a set X is called *compatible* with a binary relation Q on X if P is compatible both from the right and from the left with Q.

Theorem 6. Relation << in a projective arithmetic $\boldsymbol{A} = (M ; + , \circ, \leq)$ is compatible with the order $\leq$ in M.

Remark 3. Results of Theorems 5 and 6 are not always true for the relation <<< as the following example demonstrates.

Example 4. Let us take as a generator for a projective arithmetic the following function:

$$f_{\mathrm{T}}(n) = \begin{cases} 2^{2^n} & \text{for } n = 1, 2, 3, 4 ; \\ 2^{n+12} & \text{when } n > 4 \end{cases}$$

Then we have $2_\mu \circ 3_\mu = f^{\mathrm{T}}(f_{\mathrm{T}}(2) \cdot f_{\mathrm{T}}(3)) = f^{\mathrm{T}}(2^{2^2} \cdot 2^{2^3}) = f^{\mathrm{T}}(2^2 \cdot 2^2) = f^{\mathrm{T}}(2^2 \cdot 2^2) < 4_\mu = f^{\mathrm{T}}(2^{2^4}) = f^{\mathrm{T}}(2^2 \cdot 2^2)$. Thus, $2_\mu \circ 3_\mu = 3_\mu$ and $2_\mu <<< 3_\mu$.

At the same time, we have $2_\mu \circ 5_\mu = f^{\mathrm{T}}(f_{\mathrm{T}}(2) \cdot f_{\mathrm{T}}(3)) = f^{\mathrm{T}}(2^{2^2} \cdot 2^{17}) = f^{\mathrm{T}}(2^4 \cdot 2^{17}) = f^{\mathrm{T}}(2^{21}) = 9_\mu$. Consequently, it is not true that $2_\mu <<< 5_\mu$ although $2_\mu < 3_\mu < 5_\mu$. It means that the relation <<< is not compatible from the right with the relation <.

However, assuming that the generator $f(x)$ is a monotonous function, we have the following result.

Theorem 7. Relation <<< in a projective arithmetic $\boldsymbol{A} = (M ; + , \circ, \leq)$ is compatible from the left with the order $\leq$ in M.

Let $\boldsymbol{A} = (M ; + , \circ)$ be a projective pre-arithmetic, for which the second condition from the definition 4 is true and $f_T(1_\mu) = 1$.

Theorem 8. For any $n \in N$ and any $a_\mu , b_\mu \in \boldsymbol{A}$, from $1_\mu , 1_\mu , \dots , 1_\mu <<_n a_\mu$ follows $1_\mu , 1_\mu , \dots , 1_\mu <<_n b_\mu$ if and only if for any elements a and b from M from $a \leq b$ it follows $f_T (Sa) - f_T (a) \leq f_T (Sb) - f_T (b)$.

5. Conclusion

In the paper, explicit constructions for pre-arthmetics and non-Diophantine arithmetics are gives. Properties of such arithmetics are studied. Such properties can be related to the basic features of nature. Some physicists (cf., for example, (Zeldovich, *at al*, 1990)) emphasized that fundamental problems of modern physics are dependent on our ways of counting. This idea correlates with problems of modern physical theories in which physical systems are described by chaotic processes. Taking into account the fact that chaotic solutions are obtained by computations, physicists ask (Cartwrite and Piro, 1992; Gontar, 1997) whether chaotic solutions of the differential equations, which model different physical systems, reflect the dynamic laws of nature represented by these equations or whether they are solely the result of an extreme sensitivity of these solutions to numerical procedures and computational errors.

It is even more clear that properties of non-Diophantine arithmetics, which reflect the way people count, influence functioning of economy and are important for economical models (cf., for example, (Tolpygo, 1997)). Thus, it would be useful to build models of economical systems and processes based not on the Diophantine arithmetic but on an appropriate non-Diophantine arithmetic.

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